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PIECEWISE POWER LAW HARDENING FOR DUCTILE TEARING INSTABILITY ANALYSIS

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Abstract—A J-integral based tearing modulus approach is utilized for investigating crack growth instability in an elastic–plastic strain hardening material. A center-cracked panel of finite dimensions with linear elastic-piecewise power law hardening material behavior is analyzed under displacement controlled loading. Plane stress as well as plane strain situations are considered. Numerical results are presented for a linear elastic-piecewise power law representation and are compared with an alternate Ramberg–Osgood material representation. Stability of crack growth for a range of crack length to width ratio, a/b, of the panel is discussed via the tearing modulus parameter, T.

INTRODUCTION

In recent years the concepts of fracture mechanics have been increasingly applied in the assessment of structural integrity of power generating systems. The methods of linear elastic fracture mechanics (LEFM) have been found very useful in assessing the safety of structural components loaded in the elastic regime where crack tip deformations are confined to small regions. These methods have been successfully applied primarily in the lower shelf and the lower transition temperature range. Moreover, these methods have been often applied only to the onset of crack growth at which point fracture is assumed to have occurred.

Significant plastic zones are encountered in many practical applications involving ductile materials in the upper shelf regime. Of several fracture criteria that have been proposed to handle situations involving moderate to large scale yielding, the J-integral approach[1–3] to characterize fracture has become increasingly popular. It has been shown that the J-integral, which is based on the deformation theory of plasticity, can be regarded as a measure of the intensity of the strain and stress field surrounding a crack tip[4, 5]. The J-integral reduces to the energy release rate, G, in the linear elastic regime.

Tough and ductile nuclear reactor materials undergo extensive plastic deformations preceding initiation of crack growth. Moreover, crack growth instability is generally preceded by some amount of stable crack growth. Consequently, application of LEFM to tough and ductile materials, using crack initiation as the fracture criterion, gives considerable underestimate of their strength. The extent by which the strength is underestimated is in general dependent upon the crack growth resistance of the material, flaw size, and the manner in which the structure is loaded, i.e. displacement control or load control.

Concern to utilize the additional load carrying capability, which would otherwise lead to premature shutdown, replacement, or prohibitive maintenance costs, led to the recent attention to the resistance curve approach. This paper concerns with the J-integral-Tearing Modulus approach proposed by Paris *et al.*[6]. This approach is based on the J-integral and utilizes a J-resistance curve, similar to the R-curve approach in LEFM. A particularly attractive feature of this approach is that it includes the system supporting the cracked component for determining instability. The theoretical justification for the application of the J-integral to crack growth, and the conditions for J-controlled growth have been discussed recently[7, 8].

In this paper, stability of crack growth in a center-cracked panel under a mode I loading situation is discussed. Studies on crack growth instability of center-cracked panels obeying Ramberg–Osgood stress–strain behavior and fully plastic power law hardening behavior have been reported elsewhere[7–10]. The interest in this paper is to consider a linear elastic-piecewise power law hardening material for the panel. The reason for assuming such a material behavior is that it reasonably characterizes the stress–strain curve for most structural materials, and may be preferable to use in situations where Ramberg–Osgood representations is not convenient. In this work, the estimation procedure of [11–13] will be utilized to investigate the stability of small amounts of crack growth under J-controlled growth conditions.

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DUCTILE TEARING INSTABILITY APPROACH

An approach to stability of ductile tearing mode of crack growth called the "Tearing Modulus Approach" was proposed by Paris *et al.*[6]. This approach is based on the use of the J-integral as a crack tip characterizing parameter and a J-resistance curve. For a given load, σ, crack length, a, and other relevant dimensions of a cracked body, J may be thought of as the applied J or imposed field of strains and stresses in the crack tip region. The J-resistance curve, on the other hand, is an experimentally derived curve of J_{matl} vs crack growth, Δa. The J_{matl} is the J value a material may sustain at a crack tip for a given amount of crack growth. J_{matl} is generally considered to be a function of crack growth, Δa, at least for a given temperature and constraint at the crack tip. Hence, equilibrium considerations require

$$J(\sigma, a, c, B) = J_{\text{matl}}(\Delta a), \tag{1}$$

where, as shown in Fig. 1, c is the width of the uncracked ligament and B is the thickness of the cracked body.

The stability of crack growth is examined by comparing the calculated $\mathrm{d}J/\mathrm{d}a$ to J-resistance curve slopes in the following manner:

$$\text{(i)}\quad \text{if } \frac{\mathrm{d}J}{\mathrm{d}a} < \frac{\mathrm{d}J_{\text{matl}}}{\mathrm{d}a} \quad \text{then stable,} \tag{2}$$

or

$$\text{(ii)}\quad \text{if } \frac{\mathrm{d}J}{\mathrm{d}a} \geq \frac{\mathrm{d}J_{\text{matl}}}{\mathrm{d}a} \quad \text{then unstable.} \tag{3}$$

Paris *et al.*[6] introduced a non-dimensional parameter called the Tearing Modulus, T, which is defined as

$$T = \frac{\mathrm{d}J}{\mathrm{d}a} \cdot \frac{E}{\sigma_0^2}, \tag{4}$$

where E is the elastic modulus and σ_0 is the flow stress. The instability criterion, eqn (3) then takes the form

$$T_{\text{appl.}} > T_{\text{matl}}. \tag{5}$$

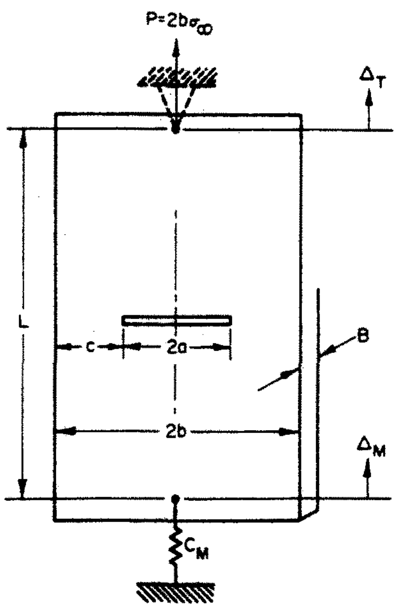

Fig. 1. A center-cracked panel with a linear elastic spring.

Crack extension invariably involves unloading behind the crack tip and some non-proportional loading. This limits the use of the J-integral since it is based on the deformation theory of plasticity. In order for a J-integral based approach to be valid, Hutchinson and Paris[7] argued that dominantly proportional strains must exist surrounding the advancing crack tip. They argued that for small amounts of crack growth a sufficient requirement for J-controlled growth is that the parameter ω

$$\omega \equiv \frac{\mathrm{d}J}{\mathrm{d}a} \cdot \frac{c}{J} \gg 1. \tag{6}$$

Further discussions on this aspect can be found in [7, 8, 14–17]. In this paper it is assumed that the J-controlled growth conditions are applicable. In the analysis that follows, relationships for $\mathrm{d}J/\mathrm{d}a$ or $T_{\text{appl.}}$ will be developed for a center-cracked panel subjected to mode I loading. A linear elastic-piecewise power law hardening material is assumed for the panel. Furthermore, only plane problems, i.e. plane stress or plane strain, are considered.

THE CENTER-CRACKED PANEL

Consider a pure power law hardening material for which the stress–strain relationship in simple tension is of the form

$$(\epsilon/\epsilon_0) = \alpha(\sigma/\sigma_0)^n, \tag{7}$$

where σ_0 and ϵ_0 are a reference stress and strain, respectively. α is a constant and n is the strain hardening index of the material. For problems with above material representation involving a single load or displacement parameter which increases monotonically, it has been shown by Ilyushin[18] that the quantities such as stress, strain and displacement increase in direct proportion to the load or displacement parameter to a certain power. For example, if σ_∞, the uniform applied stress at the ends of the panel, is the load parameter then the stress at every point increases in proportion to σ_∞, whereas the strain increases in proportion to $(\sigma_\infty)^n$.

Noting this simple dependence, Hutchinson *et al.*[12, 19] proposed that the J-integral can be expressed explicitly in terms of two functions: one, a function that depends solely on the load parameter; and second, a function that depends on the geometry of the cracked configuration. For a center-cracked panel, the J-integral can be expressed in simple and convenient functional form as follows[8, 10, 19]

$$J = \alpha b \sigma_0 \epsilon_0 F_{1n}(a/b, n) \cdot (P/P_0)^{n+1}, \tag{8}$$

where (P/P_0) is a non-dimensionalized load parameter and $F_{1n}(a/b, n)$ is a function that depends upon crack length to width ratio, a/b, and the strain hardening exponent, n. P and P_0 are the loads per unit thickness and are related to to the applied stress, σ_∞ (see Fig. 1), and the net section stress corresponding to the perfectly plastic limit, $\bar{\sigma}_L$, by

$$P = 2b\sigma_\infty \qquad P_0 \approx 2c\bar{\sigma}_L \qquad \text{with} \quad \bar{\sigma}_L = \chi\sigma_0. \tag{9}$$

χ has the value of 1 or $2/\sqrt{3}$ for plane stress or plane strain respectively. The values of the non-dimensional function F_{1n} in eqn (8) were derived from the numerical results of Refs. [12, 15] and are tabulated in [10].

An interesting feature of the solution for crack problems based on eqn (7) is that when $n = 1$, it gives the linear elastic solution whereas for $n = \infty$, it gives the rigid/perfectly plastic limiting case. In general, for an elastic-plastic loading, these two limiting solutions can be utilized to interpolate over the entire range of yielding[11].

Consider alternatively a linear elastic-piecewise power law hardening material. The stress–strain relation in simple tension is

$$(\epsilon/\epsilon_0) = (\sigma/\sigma_0) \quad \text{for } \sigma \le \sigma_0,$$

and

$$(\epsilon/\epsilon_0) = (\sigma/\sigma_0)^n \quad \text{for } \sigma > \sigma_0, \tag{10}$$

where σ_0 and ϵ_0 are related to each other by $\sigma_0 = E\epsilon_0$ and E is the elastic modulus. It should be noted that the relations in eqn (10) are special cases of eqn (7), thus allowing special cases of eqn (8) for the J-integral to be utilized for the piecewise power law hardening material behavior.

Following the estimation procedure proposed in [12] which is similar in a number of respects to that of Bucci *et al.*[11], the J-integral for a material obeying relationship in eqn (10) may be written as[13]

$$J = b\sigma_0\epsilon_0\psi_1(a/b, P/P_0)\cdot(P/P_0)^2 \quad \text{for } P \le P_0,$$

and

$$J = b\sigma_0\epsilon_0[\psi_1(a/b, P/P_0 = 1) + F_{1n}(a/b, n)\cdot\{(P/P_0)^{n+1} - 1\}] \quad \text{for } P > P_0, \tag{11}$$

where $\psi_1(a/b, P/P_0)$ and $F_{1n}(a/b, n)$ are non-dimensional functions. These functions depend on the plane stress or plane strain condition. The values of these functions are tabulated in Ref. [10] for a wide range of a/b and n. Sample curves of $\psi_1(a/b, P/P_0)$ and $F_{1n}(a/b, n)$ are shown in Figs. 2 and 3 for $n = 5$ for the plane stress condition. The function $\psi_1(a/b, P/P_0)$ is a function of (a/b) but also of the load parameter (P/P_0) which appears due to the plastic zone adjustment to physical crack length in the linear elastic solution. The load parameter (P/P_0) reflects the extent of yielding in the remaining ligaments. When $(P/P_0) = 1$, the remaining ligaments of the panel are considered to have been fully yielded.

In a manner similar to the estimation of J, the load-point displacement due to presence of a

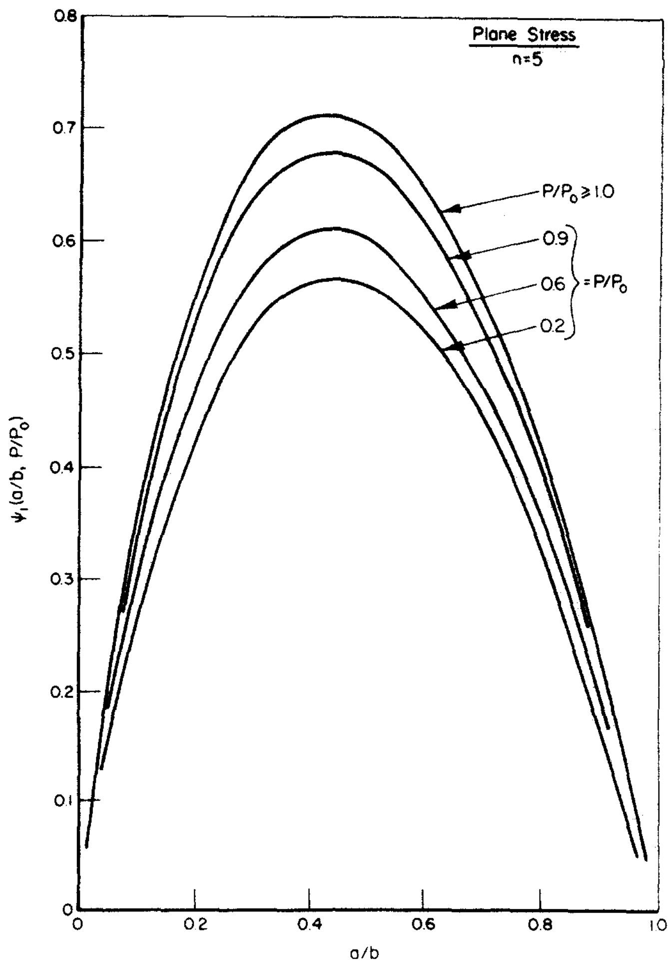

Fig. 2. Plane stress ψ_1 vs a/b for various values of P/P_0 for $n = 5$.

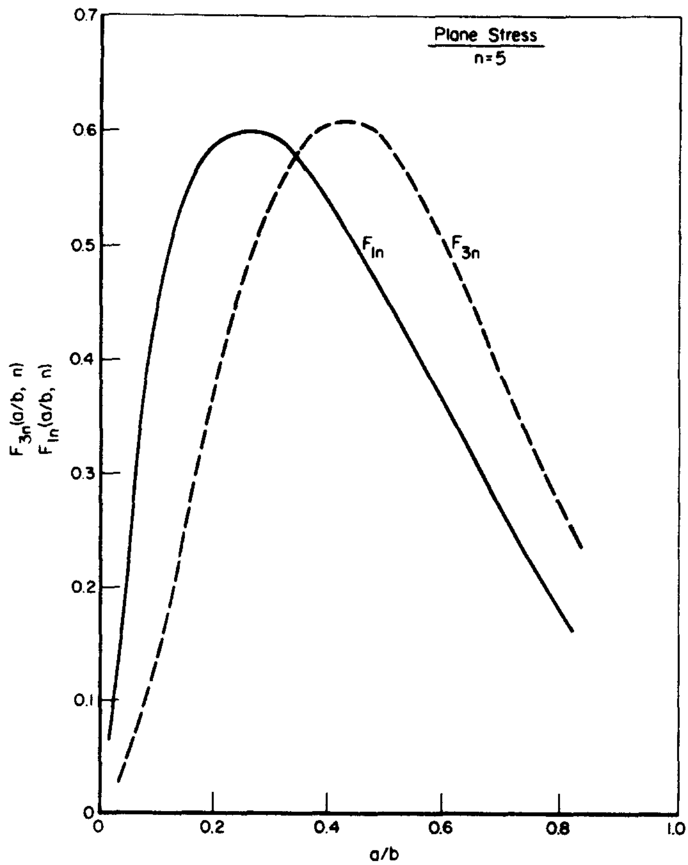

Fig. 3. Plane stress F_{1n} and F_{3n} for the strain hardening index $n = 5$.

crack, Δ_c, can be written as

$$\Delta_c = b\epsilon_0\psi_3(a/b, P/P_0)\cdot(P/P_0) \quad \text{for } P \le P_0,$$

and

$$\Delta_c = b\epsilon_0[\psi_3(a/b, P/P_0 = 1) + F_{3n}(a/b, n)\cdot\{(P/P_0)^n - 1\}] \quad \text{for } P > P_0, \tag{12}$$

where $\psi_3(a/b, P/P_0)$ and $F_{3n}(a/b, n)$ are non-dimensional functions and are tabulated in Ref. [10]. These functions are shown in Figs. 3 and 4 for $n = 5$ for the plane stress condition.

ANALYSIS FOR STABILITY OF CRACK GROWTH

To investigate crack growth instability in center-cracked panels, a linear spring in series with the panel is considered as shown in Fig. 1. The spring represents the compliance of supporting structure. A cracked component in a structural application is likely to be connected to relatively rigid structure whereupon the nature of loading will be displacement controlled. The total load-point displacement, Δ_T, consists of contributions from the cracked panel and the spring. Thus

$$\Delta_T = \Delta_{elnc} + \Delta_c + \Delta_M, \tag{13}$$

where Δ_{elnc} is the elastic contribution to the load-point displacement without the crack. Δ_c is given by eqn (12) and $\Delta_M = BC_MP$ with C_M as compliance of the linear spring.

Now, during crack growth under displacement controlled loading, the change in total displacement, Δ_T, will be zero. Thus

$$\frac{d\Delta_T}{da} = 0 = \frac{d\Delta_{elnc}}{da} + \frac{d\Delta_c}{da} + \frac{d\Delta_M}{da} \tag{14}$$

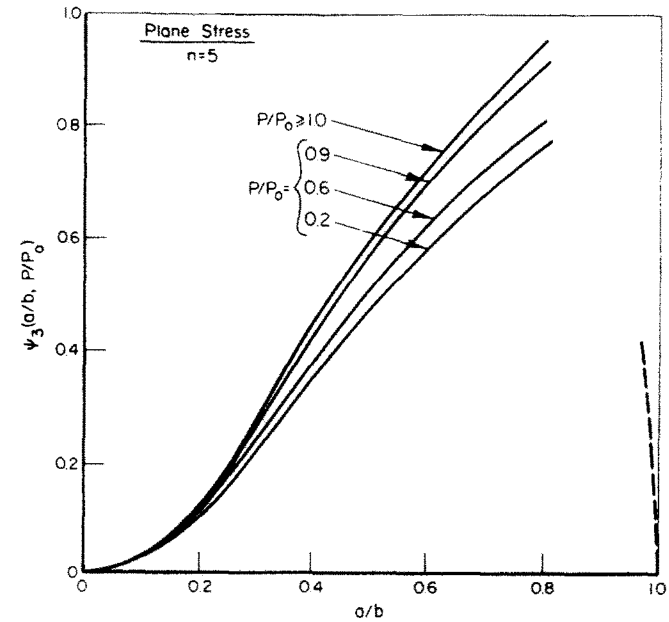

Fig. 4. Plane stress ψ_3 vs a/b for various values of P/P_0 for $n = 5$.

omitting details, the applied dJ/da using eqn (14) is [8, 10]

$$\left.\frac{\mathrm{d}J}{\mathrm{d}a}\right\|_{\Delta_T\,\text{fixed}} = \left[\left(\frac{\partial J}{\partial a}\right)_P - \left\{\left(\frac{\partial \Delta_c}{\partial a}\right)_P \left(\frac{\partial J}{\partial P}\right)_a \Big/ \left[BC_M + \left(\frac{\partial \Delta_{elnc}}{\partial P}\right)_a + \left(\frac{\partial \Delta_c}{\partial P}\right)_a\right]\right\}\right] \quad (15).$$

All the quantities in eqn (15) can be evaluated by utilizing eqns (11) and (12) and by noting that $\Delta_{elnc} = PL/2Eb$.

Equations (15) and (5), after some algebraic manipulation, give the following two conditions for instability of crack growth:

(a) for $P \leq P_0$

$$T_{\text{matl}} < \left[\frac{H_1 \cdot (P/P_0)^2}{\chi(1 - a/b)(L/b + 2BEC_M) + \psi_3(1 + \phi_2)}\right], \quad (16)$$

where

$$H_1 = \psi_1\left[\chi(1 - a/b)(L/b + 2BEC_M)\left\{\phi_3 + \frac{2}{(1 - a/b)}\right\} + \left\{(1 + \phi_1)(\phi_3 - \phi_4) - \phi_4 + \frac{\phi_1}{(1 - a/b)}\right\}\psi_3\right],$$

(b) for $P > P_0$

$$T_{\text{matl}} < \left[\frac{A_1 + A_2(P/P_0)^{n-1} + A_3(P/P_0)^n + A_4(P/P_0)^{n+1} + A_5(P/P_0)^{2n}}{\chi(1 - a/b)(L/b + 2BEC_M) + nF_{3n}(P/P_0)^{n-1}}\right], \quad (17)$$

where A_1 to A_5 and ϕ_1, ϕ_3, ϕ_4 are functions of ψ_1, ψ_3, F_{1n} and F_{1n} and their derivatives. These are defined in Appendix 1.

Equation (16) is used when $\sigma < \sigma_0$ in eqn (10), i.e. when the panel is loaded such that remaining ligaments are below full yielding ($P/P_0 \leq 1$). Thus, eqn (16) can be used in the linear elastic regime and in confined yielding situations. Equation (17) is used only when panel is loaded such that the remaining ligaments are above full yielding ($P/P_0 > 1$). In both these cases $\chi = 1$ for the plane stress panel and all functions take their plane stress values when plane stress problem is investigated for crack instability. Similarly, $\chi = 2/\sqrt{3}$ for the plane strain problem with all the functions taking their plane strain values in eqns (16) and (17).

NUMERICAL RESULTS AND DISCUSSION

Figure 5 shows the stress–strain curves for linear elastic-piecewise power law hardening and Ramberg–Osgood representations for $\alpha = 1$ and the material hardening index $n = 5$. The Ramberg–Osgood representation gives a non-linear curve even when $\sigma/\sigma_0 < 1$. The effect of this non-linear representation on $T_{\text{appl.}}$ values will be discussed and a comparison will be made with the linear elastic representation.

Figure 6 shows the $T_{\text{appl.}}$ vs (P/P_0) curves for a plane strain center-cracked panel with $a/b = 0.5$ and $(L/b + 2BEC_M) = 20$ for the two material representations. The equivalent length parameter $(L/b + 2BEC_M) = 20$ is typical of laboratory tests. As seen, the Ramberg–Osgood

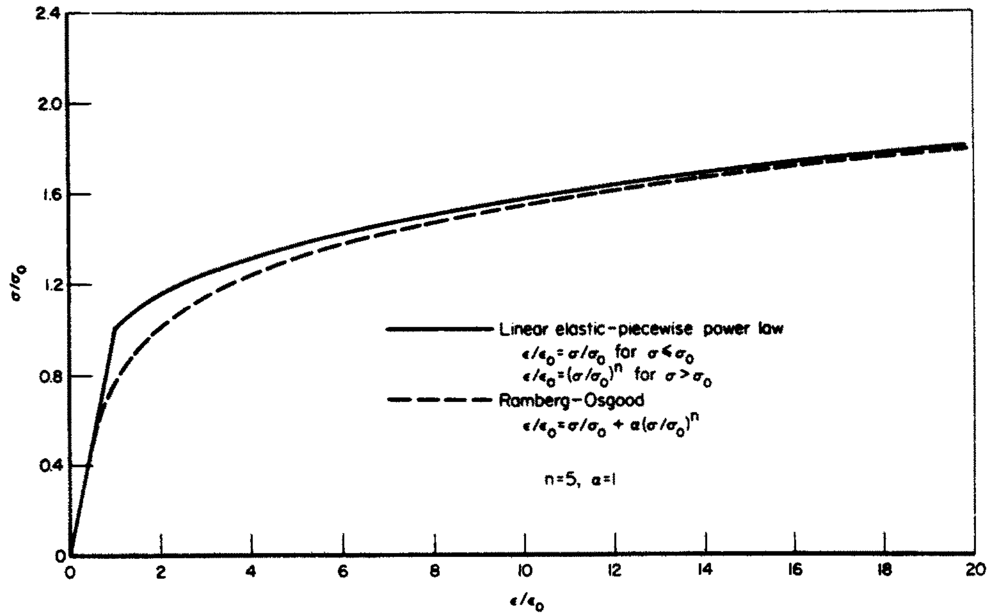

Fig. 5. Non-dimensionalized stress–strain curves for two types of material representation.

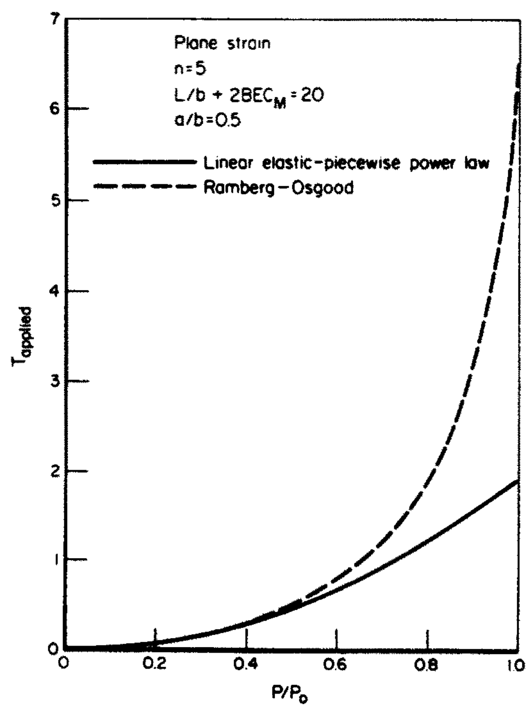

Fig. 6. $T_{\text{appl.}}$ vs P/P_0 for a plane strain panel.

representation[10] gives higher values of $T_{appl.}$ when compared with the linear elastic representation over the entire range of loading shown on Fig. 6. When $(P/P_0) \leq 1$, the Ramberg–Osgood representation[10] overestimates the value of $T_{appl.}$ by a factor of 3 or more when compared with the linear elastic representation. As discussed earlier, the tearing modulus approach requires both the J and $T_{appl.}$ for assessing instability. This information is plotted in Fig. 7 for the same conditions of Fig. 6. On such a plot the apparently large difference in $T_{appl.}$ values from the two representations tend to disappear when compared for the same $J/b\sigma_0\epsilon_0$.

Figure 8 shows a comparison of $T_{appl.}$ in the fully plastic regime, i.e. $P/P_0 > 1$. As expected, the two representations give almost the same value of $T_{appl.}$ for a given (P/P_0). The values of $T_{appl.}$ for the Ramberg–Osgood representation shown on this figure were calculated using analysis equations of Ref. [10]. It must be pointed out that for linear elastic-piecewise power law representation $T_{appl.}$ vs (P/P_0) curves are expected to be discontinuous at $P/P_0 = 1$. This is because the slope of the stress–strain curve with linear elastic-piecewise power low representation is discontinuous at $P/P_0 = 1$. Further discussions on this aspect may be found in Ref. [20].

All the numerical results presented thus far were for $a/b = 0.5$. It would be interesting to investigate the variation of $T_{appl.}$ with crack length or a/b in this case. Figure 9 shows curves of $T_{appl.}$ vs a/b for several P/P_0 values. These curves were obtained by utilizing eqns (16) and (17) for a plane strain panel with $n = 5$ and $(L/b + 2BEC_M) = 20$. It is seen that the $T_{appl.}$ decreases with the increase in a/b for the same applied load. This same information is plotted on Fig. 10 in the form of $T_{appl.}$ vs $J/b\sigma_0\epsilon_0$. Here again, $T_{appl.}$ values decrease with increase in a/b for the same $J/b\sigma_0\epsilon_0$. For determining instability of crack growth, the material values of J and their corresponding T_{matl}, which is obtained from an appropriate J-resistance curve, can be plotted on Fig. 10[21]. The intersection of the applied (analysis) and the material curves will then defined a critical value of J for a given (a/b) at which crack growth would be unstable. Knowing the critical value of J, the critical value of load parameter (P/P_0) can be calculated by utilizing the relationship in eqn (11).

The analysis presented here assumes that any crack growth, preceding instability, occurs under a J-controlled situation, and that the amount of crack growth is small. Recent work by several authors[14] indicate that, in a center-cracked panel, J-controlled growth can be justified if crack growth is less than 2% of the uncracked ligament. For other standard crack configura-

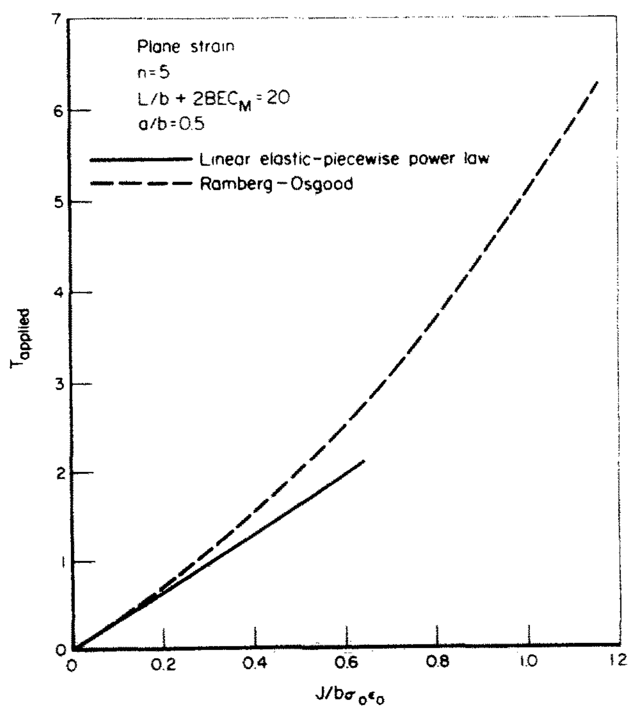

Fig. 7. $T_{appl.}$ vs $J/b\sigma_0\epsilon_0$ for a plane strain panel.

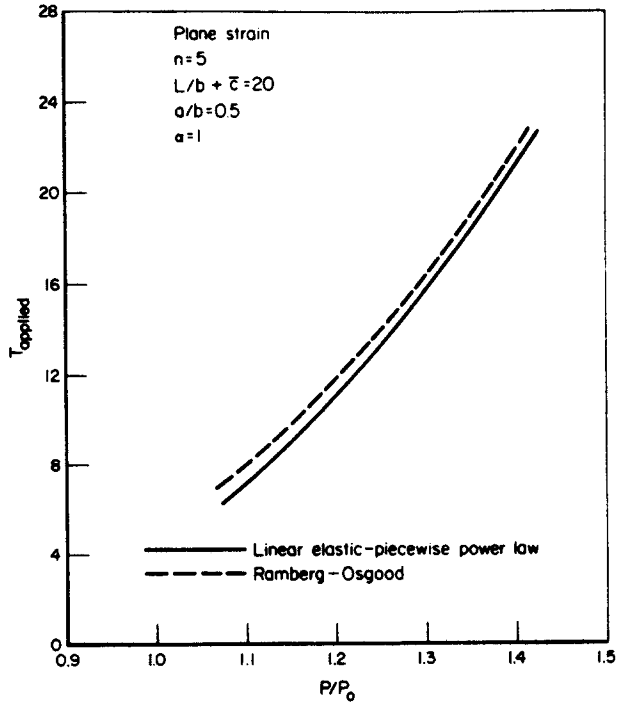

Fig. 8. A comparison of $T_{appl.}$ for a plane strain center-cracked panel in the fully plastic regime for two different material representations.

tions, a different amount of crack growth may be justifiable. For example, in compact specimens crack growth up to about 6% of the uncracked ligament can be allowed within the limitations of J-controlled growth. Because of the limitations of J-controlled growth for assessing crack instability, a more general fracture parameter is required which can handle larger amounts of crack growth.

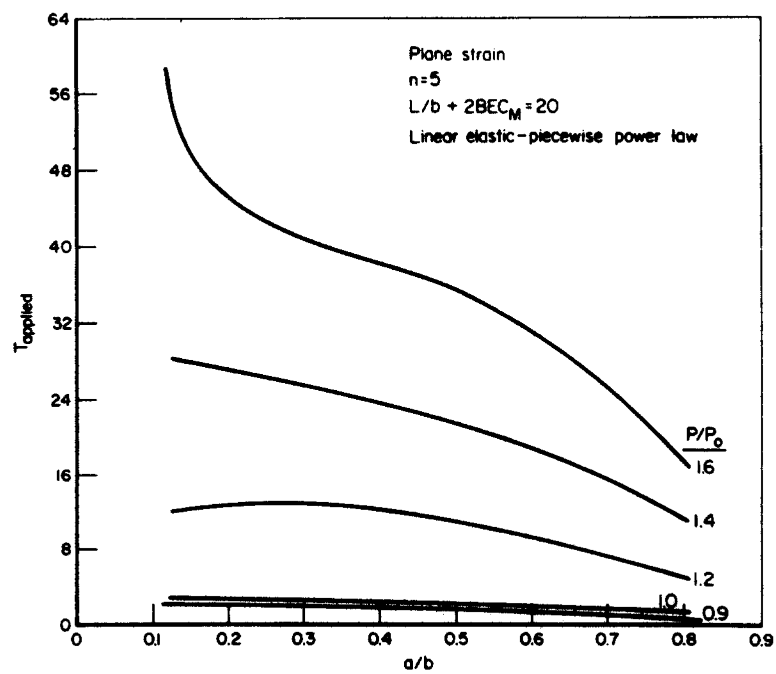

Fig. 9. $T_{appl.}$ vs a/b for several values of the load parameter (P/P_0).

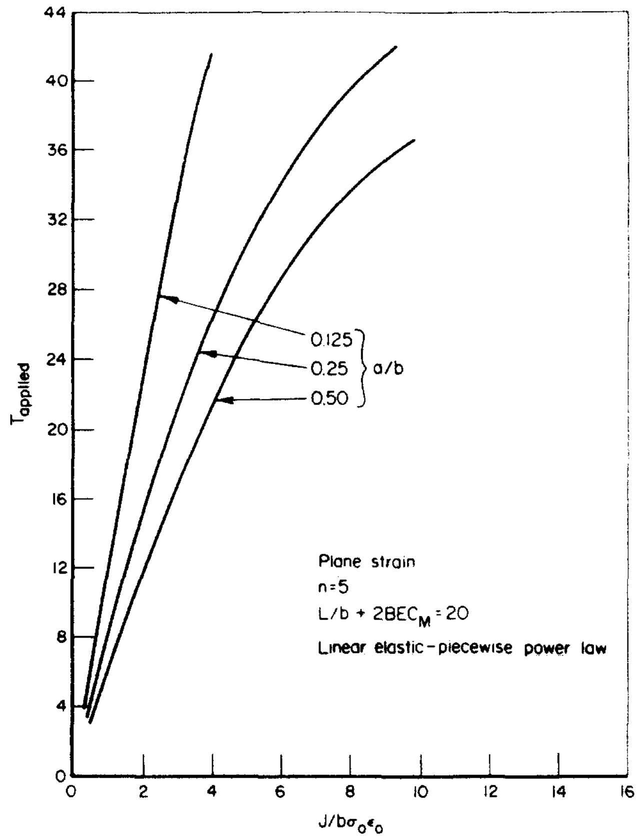

Fig. 10. $T_{appl.}$ vs $J/b\sigma_0\epsilon_0$ for a plane strain center-cracked panel obeying linear elastic-piecewise power law material behavior.

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APPENDIX 1

The functions appearing in eqns (16) and (17) are defined below:

$$
\begin{aligned}
A_1 &= \chi(1 - a/b)(L/b + \bar{c})(\phi_3\psi_1 - \phi_5 F_{1n}) \\
A_2 &= (\phi_3\psi_1 - \phi_5 F_{1n}) n F_{3n} \\
A_3 &= (\phi_6 F_{3n} - \phi_4\psi_3)(n+1) F_{1n} \\
A_4 &= \chi(1 - a/b)(L/b + \bar{c})\left(\phi_5 + \frac{n+1}{1 - a/b}\right) F_{1n} \\
A_5 &= \{n\phi_5 - (n+1)\phi_6\} F_{1n} F_{3n}
\end{aligned}
\tag{A1}
$$

and

$$
\begin{aligned}
\phi_1 &= \frac{(P/P_0)}{\psi_1}\cdot\frac{\partial\psi_1}{\partial(P/P_0)} \\
\phi_3 &= \psi_1'/\psi_1 \\
\phi_4 &= \psi_3'/\psi_3 \\
\phi_5 &= F_{1n}'/F_{1n} \\
\phi_6 &= F_{3n}'/F_{3n} \\
\bar{c} &= 2BEC_M.
\end{aligned}
\tag{A2}
$$

In (A2) primes denote partial derivatives with respect to (a/b), e.g. $\psi_1' = \partial\psi_1/\partial(a/b)$.